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Near-Best Multivariate Approximation by Fourier Series, Chebyshev Series and Chebyshev Interpolation

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A set of results concerning goodness of approximation and convergence in norm is given for L_∞ and L_1 approximation of multivariate functions on hypercubes. Firstly the trigonometric polynomial formed by taking a partial sum of a multivariate Fourier series and the algebraic polynomials formed either by taking a partial sum of a multivariate Chebyshev series of the first kind or by interpolating at a tensor product of Chebyshev polynomial zeros are all shown to be near-best L_∞ approximations. Secondly the trigonometric and algebraic polynomials formed by taking, respectively, a partial sum of a multivariate Fourier series and a partial sum of a multivariate Chebyshev series of the second kind are both shown to be hear-best L_1 approximations. In all the cases considered, the relative distance of a near-best approximation from a corresponding best approximation is shown to be at most of the order of $\prod \log n_j$, where n_j $(j = 1, 2,..., N)$ are the respective degrees of approximation in the N individual variables. Moreover, convergence in the relevant norm is established for all the sequences of near-best approximations under consideration, subject to appropriate restrictions on the function space.

1. INTRODUCTION

If f is an element of a normed linear function space X, and f^* is an element of a subspace Y, then f^* is a near-best approximation to f within a relative distance ρ (see [1]) if

$$\|f - f^*\| \leqslant (1 + \rho)\|f - f^B\|, \tag{1}$$

where f^B is a best approximation in Y to f. In case f^* is formed from f by a projection P of X into Y, then

$$\|f - f^*\| = \|f - Pf\| \leqslant (1 + \|P\|)\|f - f^B\|$$

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(see [2]), and we have a realization of (1) with

$$\rho = \|P\|.$$

Thus Pf is near-best within a relative distance $\|P\|$.

More specifically, a near-best approximation is termed practical if ρ is acceptably small (see [3]), and in particular if $\rho \le 9$ then no more than one decimal place of accuracy is lost in taking f^* in place of f^B. In the case of a projection P, we must show that $\|P\|$ is acceptably small.

Practical near-best approximations have been established by projection methods for univariate approximation in both the L_∞ and L_1 norms. Suppose that F_n denotes the projection on the partial sum of degree n of the Fourier series, G_n and H_n denote the projections on the partial sums of degree n of the Chebyshev series of first and second kinds, respectively, and I_n denotes the projection on the polynomial of degree n that interpolates in the zeros of the Chebyshev polynomial $T_{n+1}(x)$. Then, for continuous functions, it is well known (see [2, 4, 5]) that

$$\|G_n\|_\infty = \|F_n\|_\infty = \lambda_n = \frac{1}{\pi}\int_0^\pi \left|\frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}\right| dx \tag{2}$$

and

$$\|I_n\|_\infty = \gamma_n = \frac{1}{n+1}\sum_{i=0}^{n} \cot\frac{(2i+1)\pi}{4(n+1)}. \tag{3}$$

And for absolutely integrable functions it has been shown (see [6]) that

$$\|F_n\|_1 = \lambda_n$$

and

$$\|H_n\|_1 = \lambda_{n+1}.$$

Here λ_n is the classical Lebesgue constant, and γ_n is derived from the relation

$$\gamma_n = \max_x \sum_{i=0}^{n} |l_i(x)|, \tag{4}$$

where

$$l_i(x) = \prod_{k\neq i}\left(\frac{x-x_k}{x_i-x_k}\right) \tag{5}$$

and $\{x_k\}$ are the zeros of $T_{n+1}(x)$.

The constants λ_n and γ_n are known (see [4, 5]) to have asymptotic behaviours

$$\lambda_n = \frac{4}{\pi^2}\log n + O(1) \tag{6}$$

and

$$\gamma_n = \frac{2}{\pi} \log n + O(1), \tag{7}$$

and both increase so slowly with n that they do not exceed 3 for n up to 20. It follows that F_n, G_n, and I_n yield practical near-best L_∞ approximations, and F_n and H_n yield practical near-best L_1 approximations.

The convergence in norm of these univariate approximations is discussed in [1]. In the L_∞ norm, converges (as $n \to \infty$) of $F_n f$, $G_n f$, and $I_n f$ is well known if f is Dini–Lipschitz continuous (or alternatively, in the case of $F_n f$ and $G_n f$, continuous and of bounded variation). And in the L_1 norm, convergence of $F_n f$ and $H_n f$ has been established if f is square integrable.

All the above results will now be generalized to multivariate functions on hypercubes.

2. L_∞ Approximation by Fourier and Chebyshev Series

Let F denote the projection of a multivariate continuous periodic function f of N variables $x_1, x_2, ..., x_N$ over the hypercube

$$\mathscr{H}_N{}^0 = [0, 2\pi]^N = \{0 \leqslant x_1, x_2, ..., x_N \leqslant 2\pi\}$$

on the partial sum of orders $n_1, n_2, ..., n_N$ in $x_1, x_2, ..., x_N$, respectively, of its Fourier series expansion. Then it is easily seen that

$$Ff = \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \cdots \sum_{k_N=-n_N}^{n_N} C_{k_1,k_2,...,k_N} e^{i(k_1x_1+\cdots+k_Nx_N)},$$

where

$$C_{k_1,k_2,...,k_N} = \left(\frac{1}{2\pi}\right)^N \int_{\mathscr{H}_N^0} f(u_1, ..., u_N)\, e^{-i(k_1u_1+\cdots+k_Nu_N)}\, du_1 \cdots du_N .$$

It follows that

$$\begin{aligned} Ff &= \left(\frac{1}{2\pi}\right)^N \int_{\mathscr{H}_N^0} f(u_1, ..., u_N) \sum_{k_1=-n_1}^{n} \cdots \sum_{k_N=-n_N}^{n} e^{-i(k_1(u_1-x_1)+\cdots+k_N(u_N-x_N))} \\ &\qquad \times du_1 \cdots du_N \\ &= \left(\frac{1}{2\pi}\right)^N \int_{\mathscr{H}_N^0} f(x_1 + t_1, ..., x_N + t_N) \sum e^{-i(k_1t_1+\cdots+k_Nt_N)}\, dt_1 \cdots dt_N \\ &= \left(\frac{1}{2\pi}\right)^N \int_{\mathscr{H}_N^0} f(x_1 + t_1, ..., x_N + t_N) \prod_{j=1}^{N} \sum_{k_j=-n_j}^{n_j} e^{-ik_jt_j}\, dt_1 \cdots dt_N \end{aligned}$$

Thus

$$Ff = \left(\frac{1}{2\pi}\right)^N \int_{\mathscr{H}_N^0} f(x_1 - t_1, \ldots, x_N - t_N) \prod_{j=1}^{N} \frac{\sin (n_j - \frac{1}{2}) t_j}{\sin \frac{1}{2} t_j} dt_1 \cdots dt_N . \tag{8}$$

Taking norms and using the definition (2) of the Lebesgue constant λ_n, we obtain

$$\| Ff \|_\infty \leqslant \| f \|_\infty \cdot \prod_{j=1}^{N} \lambda_{n_j} . \tag{9}$$

The bound (9) is clearly attained (compare the univariate case) by taking f arbitrarily close to the function

$$f(x_1, \ldots, x_n) = \operatorname{sgn} \prod_{j=1}^{N} \frac{\sin (n_j - \frac{1}{2}) x_j}{\sin \frac{1}{2} x_j} . \tag{10}$$

Hence

$$\| F \|_\infty = \Lambda(n_1, n_2, \ldots, n_N), \tag{11}$$

where

$$\Lambda = \prod_{j=1}^{N} \lambda_{n_j} \sim \left(\frac{4}{\pi^2}\right)^N \prod_{j=1}^{N} \log n_j \tag{12}$$

from (6). It follows from Section 1 above that the Fourier partial sum Ff is a near-best L_∞ approximation within a relative distance Λ of the order of $\prod \log n_j$.

A similar result may be obtained for the Chebyshev series expansion of a nonperiodic function as follows. If $g(x_1, x_2, \ldots, x_N)$ is continuous on the hypercube

$$\mathscr{H}_N = [-1, 1]^N = \{-1 \leqslant x_1, x_2, \ldots, x_N \leqslant 1\},$$

then the related function

$$f(\theta_1, \theta_2, \ldots, \theta_N) = g(\cos \theta_1, \cos \theta_2, \ldots, \cos \theta_N) \tag{13}$$

is continuous and periodic on $\mathscr{H}_N^0$. Hence, from (9) above,

$$\| Ff \|_\infty \leqslant \| f \|_\infty \cdot \Lambda(n_1, n_2, \ldots, n_N). \tag{14}$$

This bound is attained, because the function (10) is an even periodic function of the form (13) with g arbitrarily close to a continuous function. Since f is even in $\theta_1, \theta_2, \ldots, \theta_N$, it follows that Ff only has terms in $\cos k_1\theta_1 \cdot \cos k_2\theta_2 \cdots \cos k_N\theta_N$. But the Chebyshev polynomial $T_{k_j}(x_j)$ of the first kind satisfies

$$T_{k_j}(x_j) = \cos k_j\theta_j , \qquad \text{where } x_j = \cos \theta_j ,$$

and hence

$$Ff = Gg, \tag{15}$$

where G denotes the projection on the partial sum of degrees $n_1, \ldots, n_N$ in $x_1, \ldots, x_N$ of the Chebyshev series expansion of the first kind.

Since the bound (14) is attained, it follows from (11), (13), (14), and (15) that

$$\| G \|_\infty = \| F \|_\infty = \Lambda(n_1, n_2, \ldots, n_N).$$

Thus the Chebyshev series partial sum is also a near-best L_∞ approximation within a relative distance $\Lambda(n_1, \ldots, n_N)$, given by (12), of the order of $\prod \log n_j$.

From a practical point of view these results are comparable with those for univariate functions ($N = 1$) for modest dimensions of variables and degrees of polynomials. For example, Λ is less than 9 (a loss of at most one significant figure) for

$$N = 2 \quad \text{and} \quad n_1 = n_2 \leqslant 50,$$
$$N = 3 \quad \text{and} \quad n_1 = n_2 = n_3 \leqslant 6,$$
$$N = 4 \quad \text{and} \quad n_1 = \cdots = n_4 \leqslant 2.$$

If a relative distance Λ of 99 is acceptable, which corresponds to a loss of at most two singificant figures from the best approximation, then this is achieved for

$$N = 4 \quad \text{and} \quad n_1 = \cdots = n_4 \leqslant 100,$$
$$N = 5 \quad \text{and} \quad n_1 = \cdots = n_5 \leqslant 20,$$
$$N = 6 \quad \text{and} \quad n_1 = \cdots = n_6 \leqslant 8.$$

Of course, the comparison with the univariate case is not so favourable for larger numbers of variables and higher degrees. Specifically, if $n_1 = \cdots = n_N = n$, say, so that there are $p = (n + 1)^N$ terms in the expansion, then Λ is asymptotically of order $(\log n)^N$. However, a univariate expansion with p terms has a Lebesgue constant λ_{p-1} of order $\log[(n + 1)^N - 1]$, which is essentially $N \log n$ and thus significantly smaller than Λ for large n (and $N > 1$).

3. L_∞ Approximation by Chebyshev Interpolation

Let I denote the projection of a continuous function $f(x_1, \ldots, x_N)$ over $\mathscr{H}_N = [-1, 1]^N$ on the polynomial of degrees $n_1, \ldots, n_N$ which interpolates f in the tensor product $\{(x_1, \ldots, x_N)\}$ of all possible combinations of points chosen from

$$x_1 = x_1^{(i_1)} \qquad (i_1 = 0, 1, \ldots, n_1),$$
$$x_2 = x_2^{(i_2)} \qquad (i_2 = 0, 1, \ldots, n_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_N = x_N^{(i_N)} \qquad (i_N = 0, 1, \ldots, n_N).$$

Then

$$If = \sum_{i_1=0}^{n_1} \cdots \sum_{i_N=0}^{n_N} \ell_{i_1}^{(1)}(x_1) \cdots \ell_{i_N}^{(N)}(x_N)\, f(x_1^{(i_1)}, \ldots, x_N^{(i_N)}), \tag{16}$$

where

$$\ell_i^{(j)}(x_j) = \prod_{\substack{k=0 \\ k \neq i}}^{n_j} \left(\frac{x_j - x_j^{(k)}}{x_j^{(i)} - x_j^{(k)}} \right).$$

From (16),

$$|If| \leq \|f\|_\infty \sum_{i_1=0}^{n_1} \cdots \sum_{i_N=0}^{n_N} \left(\prod_{j=1}^{N} |\ell_{i_j}^{(j)}(x_j)| \right).$$

Hence

$$\|If\|_\infty \leq \|f\|_\infty \sup_{x_1,\ldots,x_N} \prod_{j=1}^{N} \left(\sum_{i=0}^{n_j} |\ell_i^{(j)}(x_j)| \right). \tag{17}$$

The bound (17) is attained (compare the univariate case) when f is chosen of norm unity such that

$$f(x_1^{(i_1)}, \ldots, x_N^{(i_N)}) = \operatorname{sgn}[\ell_{i_1}^{(1)}(x_1^*)\, \ell_{i_2}^{(2)}(x_2^*) \cdots \ell_{i_N}^{(N)}(x_N^*)],$$

where x_j^* is the point of attainment of

$$\sup_{x_j} \sum_{i=0}^{n_j} |\ell_i^{(j)}(x_j)| .$$

Thus

$$\|I\|_\infty = \sup_{x_1,\ldots,x_N} \prod_{j=1}^{N} \sum_{i=0}^{n_j} |\ell_i^{(j)}(x_j)| . \tag{18}$$

Taking $\{x_j^{(i_j)}\}$ to be the zeros of $T_{n_j+1}(x_j)$ for $j = 1, \ldots, N$, and using the definition (4) of γ_n, (18) becomes

$$\|I\|_\infty = \Gamma(n_1, n_2, \ldots, n_N), \tag{19}$$

where

$$\Gamma = \prod_{j=1}^{N} \gamma_{n_j} \sim \left(\frac{2}{\pi}\right)^N \prod_{j=1}^{N} \log n_j \tag{20}$$

by (7). Thus If is a near-best L_∞ approximation by a relative distance Γ of the same order of magnitude $\prod \log n_j$ as Λ in Section 2 above. Once again we have a practical near-best approximatoin for modest dimensions of variables and degrees of polynomials.

4. Convergence in L_∞

Each of the projections F, G, and I considered in Sections 2 and 3, when applied to a continuous function f, produces an approximation f^* which (by (1)) satisfies

$$\|f - f^*\|_\infty \leqslant (1 + \rho) \|f - f^B\|_\infty , \tag{21}$$

where

$$\rho = C_1 \prod_{j=1}^{N} \log n_j ,$$

C_1 is a constant independent of f and n_j, and f^B is a best L_∞ approximation. Now if a "partial modulus of continuity" is defined for each component of f as

$$\omega_j(t) = \sup_{|x_j - x_j^*| \leqslant t} |f(x_1, ..., x_j, ..., x_N) - f(x_1, ..., x_j^*, ..., x_N)| ,$$

then it is known (see [7]) that

$$\|f - f^B\|_\infty \leqslant C_2 \sum_{j=1}^{N} \omega_j \left(\frac{1}{n_j + 1}\right), \tag{22}$$

where C_2 is a constant independent of f and n_j.

Combining (21) and (22), and taking $\delta_j = (n_j + 1)^{-1}$, we deduce the following result:

THEOREM 4.1. *If f satisfies a Lipschitz condition of the form*

$$\sum_{j=1}^{N} \omega_j(\delta_j) \cdot \prod_{j=1}^{N} \log \delta_j \to 0 \qquad \text{as } \{\delta_j\} \to 0 \tag{23}$$

then the multivariate Fourier series of f, the multivariate Chebyshev series of f, and the multivariate polynomial interpolating f at a tensor product of Chebyshev zeros all converge in L_∞ to f as $\{n_j\} \to \infty$. (In the case of the Fourier series, f must also be periodic for convergence on the whole hypercube.)

This is slightly weaker than the corresponding theorem for univariate approximation, which involves the Dini–Lipschitz condition

$$\omega(\delta)\log\delta \to 0.$$

For if $\delta_j = \delta$ and $\omega_j(\delta) = \omega(\delta)$ for all j, then (23) becomes

$$N\omega(\delta)(\log\delta)^N \to 0,$$

and so each partial modulus of continuity in the multivariate case has to have the size of the Nth power of the modulus of continuity required in the univariate case.

For Fourier and Chebyshev series, it is also well known that a sufficient condition for L_∞ convergence in the univariate case is that f should be continuous and of bounded variation. However, this result does not appear to generalise conveniently to the multivariate case. Results for two variables are given for the Fourier series in [8] and extended to the Chebyshev series in [9]. But, in addition to a suitable bivariate bounded variation requirement, these theorems also require that a partial derivative of f should be bounded.

5. L_1 Approximation by Fourier and Chebyshev Series

From (8), the partial sum of the Fourier series of degrees $n_1, \ldots, n_N$ of an L_1-integrable (periodic) function $f(x_1, \ldots, x_N)$ satisfies

$$Ff = \left(\frac{1}{2\pi}\right)^N \int_{\mathscr{H}_N^0} f(x_1 - t_1, \ldots, x_N + t_N) \prod_{j=1}^{N} \frac{\sin(n_j - \frac{1}{2})\, t_j}{\sin \frac{1}{2} t_j}\, dt_1 \cdots dt_N .$$

Hence

$$\|Ff\|_1 = \left(\frac{1}{2\pi}\right)^N \int_{\mathscr{H}_N^0} \left| \int_{\mathscr{H}_N^0} f(x_1 + t_1, \ldots) \prod \frac{\sin(n_j - \frac{1}{2})\, t_j}{\sin \frac{1}{2} t_j}\, dt_1 \cdots \right| dx_1 \cdots$$

$$\leqslant \left(\frac{1}{2\pi}\right)^N \iint |f(x_1 + t_1, \ldots)| \prod \left| \frac{\sin(n_j - \frac{1}{2})\, t_j}{\sin \frac{1}{2} t_j} \right| dt_1 \cdots dx_1 \cdots$$

$$= \|f\|_1 \prod \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sin(n_j - \frac{1}{2})\, t_j}{\sin \frac{1}{2} t_j} \right| dt_j .$$

Thus

$$\|F\|_1 \leqslant \prod_{j=1}^{N} \lambda_{n_j} = \Lambda(n_1, \ldots, n_N) \tag{24}$$

from (2) and (12) above.

Hence Ff is a near-best L_1 approximation within a relative distance of the order of $\prod \log n_j$, and it is a practical near-best approximation for modest values of N and n_j.

Let $H = H[n_1, n_2, ..., n_N]$ denote the projection of a function on the partial sum of degrees $n_1, ..., n_N$ of its Chebyshev series of the second kind. Suppose the function $h(x_1, x_2, ..., x_N)$ is L_1-integrable on $\mathscr{H}_N$, then the function

$$f(\theta_1, \theta_2, ..., \theta_N) = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_N \cdot h(\cos \theta_1, ..., \cos \theta_N) \quad (25)$$

is periodic and L_1-integrable on $\mathscr{H}_N{}^0$. Hence, from (24),

$$\| Ff \|_1 \leqslant \| f \|_1 \Lambda(n_1, ..., n_N). \quad (26)$$

Since f is odd in $\theta_1, ..., \theta_N$, it follows that Ff only has terms in $\sin k_1\theta_1 \sin k_2\theta_2 \cdots \sin k_N\theta_N$. But the Chebyshev polynomial of the second kind satisfies

$$\sin \theta_j \cdot U_{k_j}(x_j) = \sin(k_j + 1)\, \theta_j, \qquad \text{where } x_j = \cos \theta_j,$$

and hence it follows that

$$Ff = \sin \theta_1 \cdots \sin \theta_N \cdot H[n_1 - 1, ..., n_N - 1]h. \quad (27)$$

Now

$$\begin{aligned} \| f \|_1 &= 2^N \int_{[0,\pi]^N} | f(\theta_1, ..., \theta_N) | \, d\theta_1 \cdots d\theta_N \\ &= 2^N \int_{[0,\pi]^N} | h(x_1, ..., x_N) \sin \theta_1 \cdots \sin \theta_N | \, d\theta_1 \cdots d\theta_N \\ &= 2^N \int_{\mathscr{H}_N} | h(x_1, ..., x_N) | \, dx_1 \cdots dx_N . \end{aligned}$$

Thus

$$\| f \|_1 = 2^N \cdot \| h \|_1 . \quad (28)$$

Similarly, from (27) it follows that

$$\| Ff \|_1 = 2^N \| H[n_1 - 1, ..., n_N - 1]h \|_1, \quad (29)$$

and from (26), (28), and (29) we deduce that

$$\| H \|_1 \leqslant \Lambda(n_1 + 1, n_2 + 1, ..., n_N + 1).$$

Hence Hh is a near-best L_1 approximation within a relative distance $\Lambda(n_1 + 1, ..., n_N + 1)$, which is of course again of the order of $\prod \log n_j$.

6. Convergence in L_1

If the function $f(x_1, ..., x_N)$ is assumed to be square integrable on $\mathscr{H}_N^0$, then Ff converges in L_2 to f as a consequence of the orthogonality of the Fourier series (via the multivariate form of Bessel's inequality and the Riesz–Fischer theorem). Since L_2 is a stronger norm than L_1, it immediately follows that Ff converges in L_1 to f as $\{n_i\} \to \infty$.

Now if $h(x_1, ..., x_N)$ is square integrable on $\mathscr{H}_N$, then the function

$$f(\theta_1, ..., \theta_N) = \sin\theta_1 \cdots \sin\theta_N \cdot h(\cos\theta_1, ..., \cos\theta_N)$$

is also square integrable and (compare (28) and (29))

$$\| f - Ff \|_1 = 2^N \| h - H[n_1 - 1, ..., n_N - 1]h \|_1 .$$

Since Ff converges to f in L_1, it follows that Hh converges to h in L_1. Thus for any square integrable function the Chebyshev series of the second kind converges in L_1.

References

1. J. C. Mason, Orthogonal polynomial approximation methods in numerical analysis, *in* "Approximation Theory" (Proceedings of a Symposium) (A. Talbot, Ed.), Academic Press, New York, 1970.
2. E. W. Cheney and K. H. Price, Minimal projections *in* "Approximation Theory" (Proceedings of a Symposium) (A. Talbot, Ed.), Academic Press, New York, 1970.
3. K. O. Geddes and J. C. Mason, Polynomial approximation by projections on the unit circle, *SIAM J. Numer. Anal.* **12** (1975), 111–120.
4. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
5. M. J. D. Powell, On the maximum errors of polynomial approximations defined by interpolation and least-squares criteria, *Comput. J.* **9** (1967), 404–407.
6. J. H. Freilich and J. C. Mason, Best and near-best L_1 approximations by Fourier series and Chebyshev series, *J. Approximation Theory* **4** (1971), 183–193.
7. D. C. Handscomb, "Methods of Numerical Approximation," Pergamon, Elmsford, N.Y., 1966.
8. E. W. Hobson, "The Theory of Functions of a Real Variable and the Theory of Fourier Series," Vol. 2, 2nd ed., Cambridge Univ. Press, Cambridge, 1926.
9. J. C. Mason, Chebyshev polynomial approximations for the L-membrane eigenvalue problem, *SIAM J. Appl. Math.* **15** (1967), 172–186.